Extension of the Nambu–Jona-Lasinio model at high densities and temperatures using an implicit regularization scheme

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Traditional cutoff regularization schemes of the Nambu–Jona-Lasinio model limit the applicability of the model to energy-momentum scales much below the value of the regularizing cutoff. In particular, the model cannot be used to study quark matter with Fermi momenta larger than the cutoff. In the present work an extension of the model to high temperatures and densities recently proposed by Casalbuoni, Gatto, Nardulli, and Ruggieri is used in connection with an implicit regularization scheme. This is done by making use of scaling relations of the divergent one-loop integrals that relate these integrals at different energy-momentum scales. Fixing the pion decay constant at the chiral symmetry breaking scale in the vacuum, the scaling relations predict a running coupling constant that decreases as the regularization scale increases, implementing in a schematic way the property of asymptotic freedom of quantum chromodynamics. If the regularization scale is allowed to increase with density and temperature, the coupling will decrease with density and temperature, extending in this way the applicability of the model to high densities and temperatures. These results are obtained without specifying an explicit regularization. As an illustration of the formalism, numerical results are obtained for the finite density and finite temperature quark condensate, and to the problem of color superconductivity at high quark densities and finite temperature.

PACS numbers: 12.39.-x, 24.85+p, 21.65.+f

I. INTRODUCTION

The Nambu–Jona-Lasinio (NJL) [1] model has been the prototype model for studying chiral symmetry restoration in hadronic matter at finite baryon densities ρ_B and finite temperatures T. Since the earlier applications of the model at high ρ_B and T [2, 3], an extensive and important body of work has been done in this direction – for reviews and a comprehensive list of references see Refs. [4, 5, 6, 7, 8, 9, 10]. Because the model is nonrenormalizable, the high momentum part of the model has to be regularized in a phenomenological way. The common practice has been to regularize the divergent loop amplitudes with a three-dimensional momentum cutoff $\Lambda \sim 1$ GeV, which also sets the energy-momentum scale for the validity of the model. That is, the model cannot be used for studying phenomena involving momenta running in loops larger than Λ. In particular, the model cannot be used to study quark matter at high densities $\rho_B \sim k_F^3$ with $k_F > \Lambda$, where k_F is the quark Fermi momentum.

Recently, there have been suggestions that the NJL model can have a broader range of applicability if the parameters of the model are allowed to change with temperature and density. In a study of the color flavor locked phase of quantum chromodynamics (QCD) Casalbuoni et al. [11] have shown that in order to get sensible results within the NJL model, the ultraviolet cutoff should increase with density. Similarly, Shakin and collaborators [12] in studies of hadronic correlation functions found that results in accord with lattice QCD calculations can be obtained once a significant temperature dependence of the cutoff and of the coupling G is allowed.

Casalbuoni et al. [11] make the NJL cutoff density dependent using an analogy with an ordinary solid. In an ordinary solid, there is a natural maximum phonon frequency - the Debye frequency - that increases as the lattice spacing gets smaller. In quark matter, as the density of quarks increases, the quarks get closer together and therefore the ultraviolet cutoff Λ of the NJL model should be allowed to increase correspondingly. The cutoff can be changed consistently without spoiling the predictions of the model for the chiral properties of the vacuum if the four-fermion coupling constant G of the model is allowed to change with the cutoff. In Ref. [11], the cutoff dependence of G is obtained from the the joint consideration of the divergent expressions for the pion decay constant f_π and the gap equation for the constituent quark mass M: fixing the value of f_π at 93 MeV, the divergent expression of f_π (in the chiral limit), regulated with a cutoff Λ, leads to a constituent quark mass that is Λ dependent. The use of this $M = M(\Lambda)$ into the gap equation gives rise to a coupling G that runs with Λ. Moreover, the running of the coupling is such that $G(\Lambda)$ decreases with increasing Λ. This is certainly physically motivated and is also in accord with the interpretation [2, 13] that the cutoff in the NJL model simulates - albeit in a crude way - the property of asymptotic freedom of QCD, in the sense that the coupling between quarks decreases as higher momentum scales are probed.

The aim of the present paper is to show that it is possible to extend the applicability of the NJL model to high densities and temperatures without the use of an explicit regularization of divergences. The basic motivation for avoiding an explicit regularization such as those commonly used like the three- or four-momentum cutoff, Pauli-Villars and proper-time regularizations, is that these lead to global and gauge symmetry violations, and to the breaking of causality and unitarity. Although in many situations these problems do not have great influences on the final numerical results, there are situations, however, where they do have drastic consequences, like in studies of correlation functions. Arguments have been used and tricks invented to circumvent such problems, but no satisfactory solution has been found – for a good discussion on these issues, see for example Refs. [14, 15, 16].

Our arguments are based on an implicit regularization scheme that has been originally proposed in Ref. [17] and was used in different contexts [18], including applications to the NJL model [19, 20, 21, 22]. At the one-loop approximation, there appears only two divergent integrals, one is quadratically divergent and the other is logarithmically divergent. Once the divergent integrals are assumed to be implicitly regulated, i.e. regulated without the specification an explicit regulator, they will depend implicitly on a momentum scale, that we denote by Λ, that sets the scale at which the model is valid. These integrals satisfy well-defined scaling relations, in that the integrals at some mass scale can be related to a combination of the same integrals at some another, arbitrary scale. These scaling relations allow one to express the divergent parts of the amplitudes at finite temperature and density in terms of their counterparts at zero temperature and density. Moreover, fixing f_π in the vacuum, one can derive a scaling relation for the four-fermion coupling G which gives a running $G(\Lambda)$ similar to that found in Ref. [11]. All this is achieved, we reiterate, without specifying an explicit regularization scheme.

In the next section we review the main aspects of the implicit regularization scheme [17, 18, 19, 20, 21, 22] that are most relevant for the present paper. We start specifying the Lagrangian density we are going to use and then explain the scaling relations and their use for obtaining the solution of the gap equation within this scheme. In order to make contact with results of the literature, we solve numerically this gap equation at finite temperature and density and show the results for the quark condensate as a function of T and μ. In Section III, we make use of the scaling relations to obtain the running of the coupling G of the model and compare results with Ref. [11]. In Section IV we illustrate the use of our results for obtaining the critical temperature as a function of quark chemical potential for the spin-0 two-flavor color superconducting (2SC) gap. Our conclusions and outlook are presented in Section V.

II. CUTOFF-INDEPENDENT REGULARIZATION

For the purposes of the present work, it is sufficient to consider the simplest version of the model, specified by the two-flavor $SU(2)$ Lagrangian density

$$\mathcal{L}_{NJL} = \bar{\psi}(i\,\partial\!\!\!/ - m_0)\psi + G[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\vec{\tau}\psi)^2]. \quad (1)$$

Here ψ is the quark field operator (with color and flavor indices suppressed), m_0 is the current-quark mass matrix. At the one-loop approximation, the gap equation at finite temperature T and quark chemical potential μ, is given by (for simplicity we work in the chiral limit $m_0 = 0$)

$$M = 48\,G\,M\,T\sum_n \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{(i\omega_n+\mu)^2+E(k)^2}, \quad (2)$$

with $E(k) = \sqrt{k^2+M^2}$, and $\omega_n = (2n+1)\pi T$, $n = 0, \pm1, \pm2, \cdots$ are the Matsubara frequencies. Performing the sum, one obtains

$$M = 48\,G\,M\,[iI_{quad}(M) - I(T,\mu)]\,, \quad (3)$$

where $I_{quad}(M^2)$ is the quadratically divergent integral

$$I_{quad}(M^2) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-M^2}, \quad (4)$$

and $I(T,\mu)$ is the finite integral

$$I(T,\mu) = \int \frac{d^3k}{(2\pi)^3}\,\frac{[n_-(k)+n_+(k)]}{2\,E(k)}, \quad (5)$$

with $n_\pm(k)$ being the quark and antiquark Fermi-Dirac distributions

$$n_\pm(k) = \frac{1}{e^{[E(k)\pm\mu]/T}+1}. \quad (6)$$

At this one-loop approximation, besides the quadratically divergent integral $I_{quad}(M^2)$ there appears also a logarithmically divergent integral, $I_{log}(M^2)$

$$I_{log}(M^2) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-M^2)^2}. \quad (7)$$

The pion decay constant in vacuum, for instance, is given in terms of I_{log} – see Eq. (25).

The traditional approach is to regularize the divergent integrals I_{quad} and I_{log} with a three- or four-momentum cutoff and fit the cutoff to observables. With rare exceptions – like in Refs. [23, 24, 25] – the *finite* integral $I(T,\mu_B)$ containing the Fermi-Dirac distributions is also cutoff, leaving out in this way the high-momentum components. As an alternative to this, that avoids both the use of an explicit cutoff on I_{quad} and I_{log} and on finite integrals, we assume that each integral is regularized through a unspecified distribution $f(k/\Lambda)$, where Λ is a parameter with the dimensions of momentum such that

$$I_{quad}(M^2) = \int \frac{d^4k}{(2\pi)^4}\frac{f(k/\Lambda)}{k^2-M^2}, \quad (8)$$

$$I_{log}(M^2) = \int \frac{d^4k}{(2\pi)^4}\frac{f(k/\Lambda)}{(k^2-M^2)^2}. \quad (9)$$

The parameter Λ sets the regularization scale and the momentum scale for the applicability of the model - given a value for Λ, the model should be used for studying phenomena occurring at momenta of the order or below this value. As usual, the model (in the chiral limit) has two parameters to be fixed, the coupling G and the regularization parameter Λ. The regularization parameter can be fixed by fitting, for example, I_{quad}, which, as we shall see shortly, gives the quark condensate in vacuum. Note that one does not need to fit both I_{quad} and I_{log}, since they are not independent quantities. Obviously, there is another freedom in the problem that is the form of the regularization function, in that changing the functional form of the regularization function, changes the Λ dependence of the results. More specifically, the method consists in manipulating divergent amplitudes in a way divergences are isolated in terms of I_{quad} and I_{log} that can be fitted to vacuum quantities and finite integrals are freely integrated without the regularization function.

In order to make the Λ dependence explicit, but still not specifying the regulating function, we define the dimensionless ratio $\lambda = M/\Lambda$ and write $I_{quad}(M^2)$ and $I_{log}(M^2)$ in terms of dimensionless integrals $J_{quad}(\lambda^2)$ and $J_{log}(\lambda^2)$ as

$$
\begin{aligned}
I_{quad}(M^2) &= \Lambda^2 J_{quad}(\lambda^2), && (10)\\
I_{log}(M^2) &= J_{log}(\lambda^2), && (11)
\end{aligned}
$$

with

$$
\begin{aligned}
J_{quad}(\lambda^2) &= \int \frac{d^4u}{(2\pi)^4}\frac{f(u)}{u^2-\lambda^2}, && (12)\\
J_{log}(\lambda^2) &= \int \frac{d^4u}{(2\pi)^4}\frac{f(u)}{(u^2-\lambda^2)^2}, && (13)
\end{aligned}
$$

where the integration variable u is dimensionless. Differentiating these with respect to λ^2, one obtains

$$
\begin{aligned}
\frac{\partial J_{quad}\left(\lambda^2\right)}{\partial\lambda^2} &= J_{log}\left(\lambda^2\right), && (14)\\
\frac{\partial J_{log}\left(\lambda^2\right)}{\partial\lambda^2} &= -\frac{i}{(4\pi)^2}\frac{1}{\lambda^2}. && (15)
\end{aligned}
$$

Note that in differentiating $J_{log}(\lambda^2)$ in Eq. (13) with respect to λ^2 the resulting integral is finite even in the absence of the regularization function. Therefore, as mentioned above one can remove the regularization function and freely integrate the finite integral. Eqs. (14) and (15) can be integrated, giving

$$
\begin{aligned}
iJ_{quad}(\lambda^2) &= \frac{1}{(4\pi)^2}\lambda^2\left(\log\lambda^2-1\right)+c, && (16)\\
iJ_{log}(\lambda^2) &= \frac{1}{(4\pi)^2}\log\lambda^2, && (17)
\end{aligned}
$$

where c is an integration constant given by

$$
c = 1+\frac{i}{\pi^2}\int d^4u\frac{f(u)}{u^2-1}. \qquad (18)
$$

Clearly, c reflects the extra freedom alluded above that one has in choosing the functional form of the regulation function. Different choices of f will give different c's, but all the rest is independent of the different choices. A detailed discussion on this will be presented in a separate publication [22]. Note that in principle one has another integration constant in Eq. (17), but this can be absorbed in Λ when using the divergent integrals I_{quad} and I_{log} in terms of M/Λ.

An important relation involving the divergent integrals, that will be useful for applications at finite densities and temperatures can be obtained by integrating Eqs. (14) and (15) between λ_0 and λ. This is given by

$$
\begin{aligned}
J_{quad}(\lambda^2) &= J_{quad}(\lambda_0^2)+(\lambda^2-\lambda_0^2)\,J_{log}(\lambda_0^2)\\
&+ \frac{i}{(4\pi)^2}\left[\lambda^2-\lambda_0^2-\lambda^2\log\left(\frac{\lambda^2}{\lambda_0^2}\right)\right], && (19)\\
J_{log}(\lambda^2) &= J_{log}(\lambda_0^2)-\frac{i}{(4\pi)^2}\log\left(\frac{\lambda^2}{\lambda_0^2}\right). && (20)
\end{aligned}
$$

With these one can isolate vacuum from the temperature- and density-dependent contributions in expressions for physical quantities, in particular in the gap equation. Let M_0 be the mass obtained by solving the gap equation in vacuum and M the mass in medium. Supposing that Λ is the same in vacuum and in medium, i.e. $\Lambda = \Lambda_0$, one has

$$
\begin{aligned}
I_{quad}(M^2) &= I_{quad}(M_0^2)+\left(M^2-M_0^2\right)\,I_{log}(M_0^2)\\
&+ \frac{i}{(4\pi)^2}\left[M^2-M_0^2-M^2\log\left(\frac{M^2}{M_0^2}\right)\right], && (21)\\
I_{log}(M^2) &= I_{log}(M_0^2)-\frac{i}{(4\pi)^2}\log\left(\frac{M^2}{M_0^2}\right), && (22)
\end{aligned}
$$

and the gap equation Eq. (3) can be written as

$$
\begin{aligned}
M &= 48\,G\,M\left\{-\frac{\langle\overline{\psi}\psi\rangle_0}{12M_0}-(M^2-M_0^2)\frac{f_\pi^2}{12M_0^2}\right.\\
&\left.-\frac{1}{(4\pi)^2}\left[M^2-M_0^2-M^2\log\left(\frac{M^2}{M_0^2}\right)\right]-I(T,\mu)\right\}, && (23)
\end{aligned}
$$

where we have made use of the expression for the quark condensate in vacuum

$$
\langle\overline{\psi}\psi\rangle_0 = -12M_0\,iI_{quad}(M_0^2), \qquad (24)
$$

and the expression for f_π in vacuum,

$$
f_\pi^2 = -12M_0^2\,iI_{log}(M_0^2). \qquad (25)
$$

Therefore, once vacuum values are known, the temperature- and density-dependent quark mass M can be obtained by solving Eq. (23). The corresponding expression when using a $m_0 \neq 0$ is a little more complicated than the one shown in Eq.(23), since the expression for f_π^2 contains, in addition to I_{log}, a finite integral involving the pion mass.

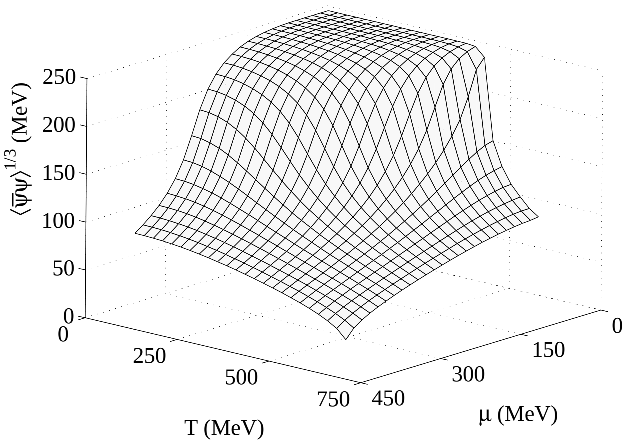


FIG. 1: One flavor quark condensate $\langle\overline{\psi}\psi\rangle$ as function of temperature T and quark chemical potential μ.

For illustrative purposes, we present in Fig. 1 the quark condensate as a function of temperature T and quark chemical potential μ – the results here are obtained solving the gap equation with $m_0 = 5.5$ MeV, $G = 4.9 \times 10^{-6}$ MeV^{-2} and the fitting mass $M_0 = 312$ MeV.

As seen in Fig. 1, this implicit regularization scheme gives the expected reduction of the quark condensate in medium. In addition, this reduction is in qualitative agreement with the result obtained with traditional cutoff regularization schemes - see for example Fig. 26 of Ref. [4]. In the chiral limit, the order of the phase transition is first order. It is also worth mentioning that with these same parameters one is able to obtain good values for the π- and σ-meson masses [22].

Of course, the point of the exercise here was not only to show that this different way of handling the divergent integrals gives results in accord with the expected phenomenology. It should be noted here is that once we have eliminated $I_{quad}(M_0^2)$ and $I_{log}(M_0^2)$ in favor of $\langle\bar{\psi}\psi\rangle_0$ and f_π we have kept Λ implicity fixed. If one wants to change Λ without changing the low-energy results G has to run with Λ, as in Ref. [11]. The point here is that one can implement this using the scaling relations of Eqs. (19) and (20), which are independent of an explicit regularization. This will be done in the next Section.

III. RUNNING OF THE COUPLING

Fixing $f_\pi = 93$ MeV and using the the expression of f_π in terms of I_{log} at the scales (M, Λ) and (M_0, Λ_0), one obtains $M = M(\Lambda)$ as the solution of the transcendental equation

$$M^2 = M_0^2 \frac{\Lambda^2}{\Lambda_0^2} \exp\left[\frac{4\pi^2}{3}\frac{f_\pi^2}{M^2}\left(\frac{M^2}{M_0^2} - 1\right)\right]. \qquad (26)$$

Using the value of $\langle\bar{\psi}\psi\rangle_0$ at the reference scale (M_0, Λ_0), and using the scaling relation of Eq. (19) to obtain I_{quad} at (M, Λ) in the gap equation, one obtains $G = G(\Lambda)$ as

$$\left[48\Lambda^2 G(\Lambda)\right]^{-1} = \frac{-\langle\bar{\psi}\psi\rangle_0}{12 M_0 \Lambda_0^2} - \left(\frac{M^2}{\Lambda^2} - \frac{M_0^2}{\Lambda_0^2}\right)\frac{f_\pi^2}{12 M_0^2} - \frac{1}{(4\pi)^2}\left[\frac{M^2}{\Lambda^2} - \frac{M_0^2}{\Lambda_0^2} - \frac{M^2}{\Lambda^2}\log\left(\frac{M^2\Lambda_0^2}{\Lambda^2 M_0^2}\right)\right], \qquad (27)$$

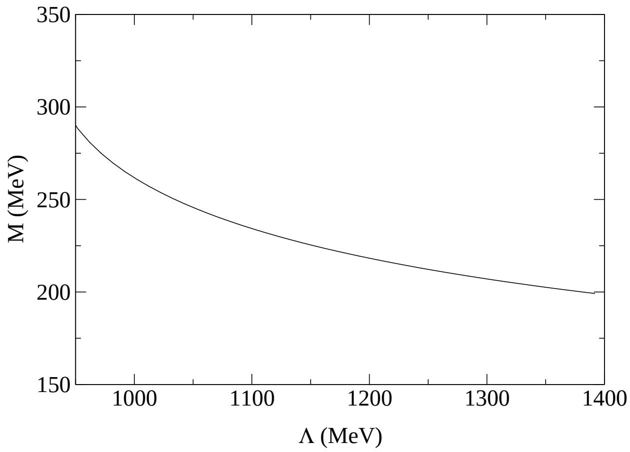
where it is understood that $M = M(\Lambda)$, as given by Eq. (26).

This is our main result in the present paper. It is an interesting result, in that it was obtained without specifying any explicit regularization, only very general scaling relations of the divergent integrals were used. In this sense, the result seems very general and robust. Once a temperature and density dependence for Λ is specified, G becomes also temperature and density dependent.

Note that the entire line of arguments can be turned around, instead of fixing f_π at some value, one could *postulate* a running behavior for $G = G(\Lambda)$ and work backwards. What would change in this case? First, using the postulated $G(\Lambda)$ in the gap equation, one would obtain a corresponding $M(\Lambda)$. When this $M(\Lambda)$ is replaced in the expression for f_π, in general f_π will be also Λ dependent, but this Λ dependence would be very weak since the integral for f_π is only logarithmically divergent. In this way, the chiral physics in vacuum would be maintained. This is very interesting, since one could use the predicted running of the QCD coupling constant for G and in this way model in a crude way the asymptotic freedom of QCD in the NJL model.

In Fig. 2 we present the numerical results for the Λ dependence of M, as obtained from the solution of Eq. (26) after solving the gap equation. The constituent mass decreases as Λ is increased. If G were kept fixed, M would increase of course. But G decreases with Λ, as shown in Fig. 3, and the net effect is that M decreases. These results for $M = M(\Lambda)$ and $G = G(\Lambda)$ are in qualitative agreement with the results of Ref. [11].

In closing this Section we reiterate that the purpose for making G to run with Λ is to extend the applicability of the model to high densities and temperatures. At high densities and temperatures, high momentum components are present in the system and a cutoff of the order of the chiral symmetry breaking scale invalidates the use of the model in such situations. In order to illustrate the use of the extension in practice we consider in the next Section color superconductivity in high-density quark matter.



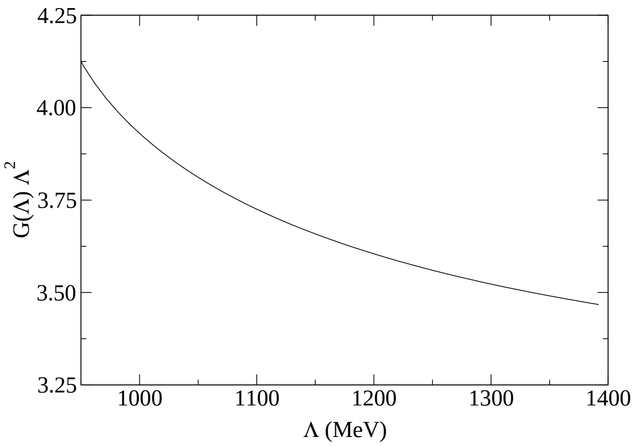


FIG. 3: Running of the coupling $G(\Lambda)$.

IV. COLOR SUPERCONDUCTIVITY

For reviews and a comprehensive list of references on the subject of color superconductivity see Refs. [26, 27, 28, 29, 30, 31, 32, 33]. Since the aim here is to illustrate the formalism we simplify matters by using the same Lagrangian density as used above, although for obtaining a better phenomenological description of both chiral symmetry breaking and color superconductivity a more general four-fermion Lagrangian should be used [34, 35, 36]. At the one-loop level the self-consistent equation for the superconducting gap is given by (we use the same letter G to denote both the diquark coupling here and the quark-antiquark coupling in the last section)

$$1 = 16\, G \int \frac{d^4k}{(2\pi)^4} \frac{i}{k_0^2 - (k+\mu)^2 - \Delta^2} + (\mu \rightarrow -\mu). \quad (28)$$

The integrals above are divergent. The application of the implicit regularization scheme to this problem proceeds as follows [21]. Instead of introducing a cutoff in the integrals, the integrands are assumed to be implicity regularized and then manipulated in a way divergences are isolated in μ-independent divergent integrals. These divergent integrals can be related to the divergent integrals I_{quad} and I_{log} of the problem of chiral symmetry breaking in vacuum though the use of the scaling relations discussed above. From the manipulation of the integrand in Eq. (28) also result finite integrals, and these are integrated without imposing any restriction to their integrands. Initially we consider $T = 0$. In this case the equation for the superconducting gap is given by

$$\begin{aligned} 1 \; = \; & 2\, G\,(\Lambda)\, \Lambda^2 \left\{ -\frac{4}{3} \frac{\langle\overline{\psi}\psi\rangle_0}{M_0\, \Lambda_0^2} - \frac{4}{3} \left(\frac{\Delta^2}{\Lambda^2} - \frac{M_0^2}{\Lambda_0^2} \right) \frac{f_\pi^2}{M_0^2} \right. \\ & - \; \frac{1}{\pi^2} \left[\frac{\Delta^2}{\Lambda^2} - \frac{M_0^2}{\Lambda_0^2} - \frac{\Delta^2}{\Lambda^2} \log \left(\frac{\Delta^2 \Lambda_0^2}{\Lambda^2 M_0^2} \right) + 2 \frac{\mu^2}{\Lambda^2} \right] \\ & \left. + \; \frac{\mu^2}{\Lambda^2} \left[\frac{8}{3} \frac{f_\pi^2}{M_0^2} - \frac{2}{\pi^2} \log \left(\frac{\Delta^2 \Lambda_0^2}{\Lambda^2 M_0^2} \right) \right] \right\}, \end{aligned} \quad (29)$$

where we used the manipulations of the integrand in Eq. (28) as explained above and shown explicitly in Ref. [21].

One very interesting result of the application of the implicit regularization to the problem of color superconductivity is that, as shown with greater detail in Ref. [21], the superconducting gap as a function of μ does not vanish at $\mu \simeq \Lambda$, as happens with the traditional cutoff schemes. The numerical result for Δ is shown by the solid line in Fig. 4. This is interesting because a nonvanishing gap at high quark densities is predicted by QCD [37] - see also the reviews in Ref. [26, 27, 28, 29, 30, 31, 32, 33]. Of course, if the implicit regularization scale Λ is kept fixed, the use of the model at high densities is questionable. However, we are able to make explicit the Λ-dependence in the gap equation (and in other physical quantities as well) by extracting the Λ dependence from the implicit regularization function. In this way, one can very easily extend the applicability of the model to larger values of μ by allowing a running $G(\Lambda)$ and a μ dependence for Λ.

For the density dependence of Λ, there is a great deal of arbitrariness. One could fix this dependence, for instance, by matching the density dependence of $G(\Lambda)\Lambda^2$ with the prediction of perturbative QCD for the running of the QCD coupling constant α_s at high densities – the one loop prediction is that the coupling decreases logarithmically with μ for large values of μ. However, for our purposes here of showing the qualitative results only, we use the simple formula for $\mu \geq \mu_0 = 235$ MeV

$$\Lambda = \Lambda_0 \left[1 + \alpha \, \log \left(\frac{\mu}{\mu_0} \right) \right], \quad (30)$$

where α is a constant. When this is used into the expression for $G(\Lambda)\Lambda^2$, one obtains that $G(\Lambda)\Lambda^2$ decreases logarithmically with μ, mocking up in a rather crude way the prediction of perturbative QCD for the running of α_s with μ. In Fig. 4 we show the numerical results Δ as a function μ for different values of α. We use for the diquark pairing strength the value $G = 3.1$ GeV^{-2}. The other parameters are $M_0 = 312$ MeV and $\Lambda_0 = 932$ MeV, which are the values obtained in vacuum for $f_\pi = 93\,\mathrm{MeV}$ and $\langle\bar\psi\psi\rangle_0 = (-\ 250.\,\mathrm{MeV})^3$. The results show the expected behavior of the gap growing faster with μ as Λ increases with μ, i.e. as α increases.

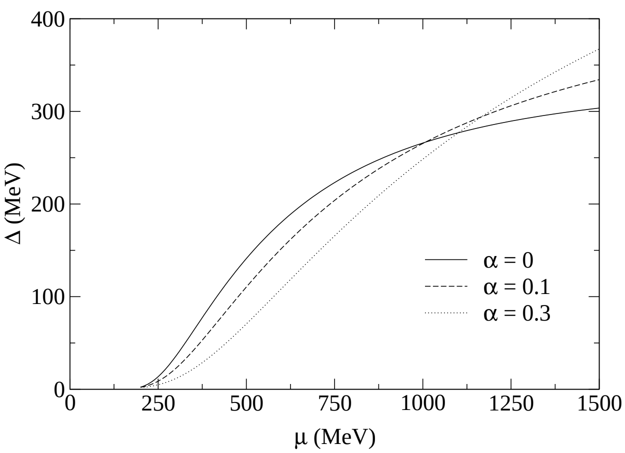


FIG. 4: Zero temperature superconducting gap as a function of quark chemical potential for three different values of α in Eq. (30).

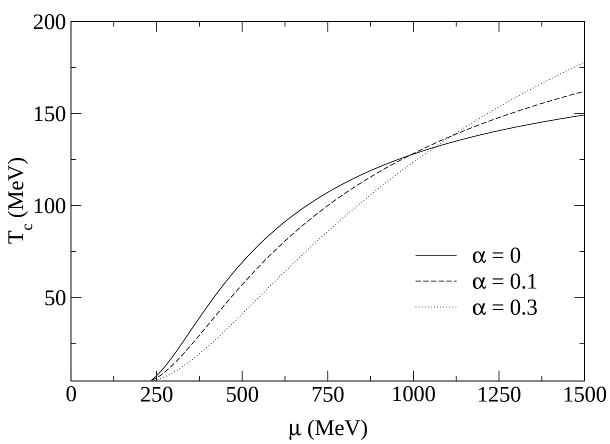


FIG. 5: Critical temperature for the superconducting gap as a function of μ for different values of α in Eq. (30).

For completeness, we include the effects of temperature. We calculate the critical temperature T_c above which $\Delta = 0$, for different values of μ. We rewrite Eq. (28) for $\Delta = 0$ including the effect of temperature

$$\begin{aligned} 1 &= 16\,G \int \frac{d^4k}{(2\pi)^4} \frac{i}{k_0^2 - (k+\mu)^2} \tanh\left(\frac{\beta}{2}|k+\mu|\right) \\ &+ (\mu \to -\mu), \end{aligned} \tag{31}$$

where

$$\tanh\left(\frac{\beta}{2}\,|k\pm\mu|\right) = 1 - 2n'_\pm(k)\,, \tag{32}$$

with

$$n'_\pm(k) = \frac{1}{e^{\beta|k\pm\mu|}+1}. \tag{33}$$

Using the same manipulations to isolate the divergent integrals as explained before and after some algebraic effort we can write

$$\begin{aligned} \frac{1}{4G} &= \Lambda^2 \Bigg\{ -\frac{2}{3}\frac{\langle\bar\psi\psi\rangle_0}{M_0\Lambda_0^2} + \frac{2}{3}\left(\frac{M_0^2}{\Lambda_0^2} + \frac{2\mu^2}{\Lambda^2}\right)\frac{f_\pi^2}{M_0^2} \\ &+ \frac{1}{2\pi^2}\left[\frac{M_0^2}{\Lambda_0^2} - \frac{3\mu^2}{\Lambda^2} - \frac{2\mu^2}{\Lambda^2}\ln\left(\frac{\mu^2}{\Lambda^2}\frac{\Lambda_0^2}{M_0^2}\right)\right]\Bigg\} \\ &+ Q(\mu,T) + Q(-\mu,T), \end{aligned} \tag{34}$$

where

$$\begin{aligned} Q(\mu,T) &= -\frac{2}{\pi^2}\int_0^\infty dk \frac{k^2}{(k^2+\mu^2)^{1/2}}\left[n'_+(k) + n'_-(k)\right] \\ &+ \frac{2\mu}{\pi^2}\int_0^\infty dk \frac{k^3}{(k^2+\mu^2)^{3/2}}\left[n'_+(k) - n'_-(k)\right] \\ &- \frac{3\mu^2}{\pi^2}\int_0^\infty dk \frac{k^4}{(k^2+\mu^2)^{5/2}}\left[n'_+(k) + n'_-(k)\right] \\ &+ \frac{1}{2\pi^2}\int_0^\infty dk\, k^2\, I(k)\, \tanh\left(\frac{\beta}{2}\,|k+\mu|\right), \end{aligned} \tag{35}$$

with

$$I(k) = \frac{2k^5 + 5k^2\mu^2(k-\mu) - 2\mu^5 - 2(k^2+\mu^2)^{5/2}}{(k^2+\mu^2)^{5/2}(k-\mu)}. \tag{36}$$

In Fig.(5) we plot the critical temperature as function of μ. In the solid line we have kept the coupling G fixed and in the dashed and dotted lines the coupling is a function of μ through the μ dependence of Λ as in Eq. (30). As seen, the critical temperature increases with μ, and this increase is faster as α increases for large μ. The values of T_c obtained within this scheme seem a little larger than obtained with cutoff regularization.

V. CONCLUSIONS AND PERSPECTIVES

We have considered the extension of the NJL model to high densities and temperatures proposed in Ref. [11] and used an implicit regularization scheme to handle ultraviolet divergences. This extension is implemented by allowing the regularization scale Λ to increase at high densities with the simultaneous decrease of the coupling G. Making use of the scaling relations of Eqs. (19) and (20), and the definitions of Eqs. (10) and (11), the two one-loop divergent integrals I_{quad} and I_{log} at scales Λ_0 and M_0 can be related to the same I_{quad} and I_{log} at different scales Λ and M.

Although our numerical results are in qualitative agreement with the ones of the Casalbuoni et al. [11]

for the μ dependence of the superconducting gap Δ, we believe the present approach generalizes their results in several aspects. Perhaps the most important one is the fact that no specific regularization scheme was used to obtain the running of G with Λ. This fact is very important in view that the commonly used regularization schemes, like those based on a three- or four-momentum cutoff, or Pauli-Villars and proper-time regularizations, lead in general to global and gauge symmetry violations, and to the breaking of causality and unitarity. In Ref. [38] hadronic correlation functions were calculated with the present implicit regularization method and shown that unitarity is preserved (thresholds are independent of Λ). In addition, it is possible to show that causality is also preserved, by checking that the amplitudes satisfy the correct dispersion relations. Gauge symmetry is also preserved at the one-loop level, as shown in Ref. [19] in the context of the gauged NJL model. A complete discussion of these matters will be discussed in detail in separated publication [22].

Another interesting aspect of the present approach is that it can be used with heavy quarks. The scaling relations involve the ratio of the quark mass to the implicit regulation scale and naturally take into account short-distance effects as the quark mass increases. This is very important in connection with studies of heavy-quark bound states in highly excited quark matter.

Acknowledgments: Work partially supported by the Brazilian agencies CAPES, CNPq and FAPESP.

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